ISSN: 2338-0128
VOLUME: 6

PROCEEDING

BaSIC 2016

THE 6th BASIC SCIENCE INTERNATIONAL CONFERENCE

BaSIC 2016
Basic Science International Conference

2nd - 3rd MARCH 2016

ATRIA HOTEL & CONFERENCE, MALANG, EAST JAVA
INDONESIA

ORGANIZED BY

FACULTY OF MATHEMATICS & SCIENCES
BRAWIJAYA UNIVERSITY

PROCEEDINGS OF THE 6th ANNUAL BASIC SCIENCE INTERNATIONAL CONFERENCE

"Enhancing Innovation in Science for Sustainable Development"
ATRIA HOTEL AND CONFERENCE, MALANG, INDONESIA
March, 2nd – 3rd 2016

Please cite the proceeding as "Proceeding of Annual Basic Science International Conference Vol. 6" with the following abbreviation:
Proc. Basic Sci., **6**

Issued on **June 7th, 2016**

FOREWORD

The 6th Annual Basic Science International Conference (BaSIC 2016) had been successfully held on 2 – 3 March 2016 at Atria Hotel, in Malang, Indonesia. The conference theme this year is "*Enhancing Innovation in Science for Sustainable Development*". The conference is aimed at promoting scientific research activities by fellow scientists in Indonesia and overseas, in the hope of building and strengthening networks and collaborations. Additionally, the conference is also designed to bring experts as well as students together from different disciplines related to basic sciences, to stimulate the formation of new collaborations. So, it is an event where new generation of scientists will coalesce with the senior and experienced ones.

We do thank all participants for their contributed talks, the keynote speakers, as well as the invited speakers for coming and sharing their knowledge with us. The presenters actively contributed in sending their articles to be published in this proceeding. We also thank Brawijaya University and Faculty of Sciences in particular, the organizing team from the Department of Mathematics, Faculty of Sciences, Brawijaya University, as well as all members of the scientific committee.

We are delighted that the proceeding of the 6th Annual Basic Science International Conference (BaSIC 2016) had been completed. It is a book containing papers that had been presented in the BaSIC conference. Moreover, the articles in this proceeding are divided into a breath of the conference subjects of Material Science and technology, Science and Technology Education, Environmental Science and Technology, Molecular and Health Science, Mathematics, Statistics, and Modeling, Instrumentation and Measurement, as well as Energy. The proceeding is aimed at collecting and sharing any useful information that had been gathered during the BaSIC conference.

The editorial team has made some editing and correction needed in some cases. Most of the editing correction are conducted and concentrated in the organization of the paper based on the guideline and the language. Some figures and tables were corrected, and placed accordingly. In addition, the language is the most time-consuming work; hence on behalf of the committee we apologize for the late publishing of this book and for any inconvenience as a result of the delay.

We give our gratitude to the reviewing and editing team for their hard work and for making the publication of this proceeding happen. We again thank all participants and presenters for the kindness to be part of the BaSIC conference. We hope the readers of this book could gain new knowledge, information, and idea for a research and further research collaboration, particularly in the topics or subjects related to basic sciences.

Best regards,

Achmad Efendi, PhD
Chairman of BaSIC 2016

WELCOME MESSAGE

On behalf of the Dean of Faculty of Mathematics and Natural Sciences, we are very pleased to welcome you in the proceeding of the Sixth Annual Basic Sciences International Conference 2016. This proceeding is one of the continuation for the conference. Based on these papers, hopefully more collaboration can be initiated or should be followed up.

I would like to express my gratitude to all of the contributed papers, also keynote and invited speakers. Many thanks also goes to the reviewers and the editorial team for the big effort in supporting this proceeding.

Last but not least my big appreciation to the steering and organizing committees, in realizing this proceeding.

Faculty of Mathematics and Natural Sciences,

Dean,

Prof. Dr. Marjono, M.Phil.

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Published online, on June 7, 2016.

Characterization of The Solution of Non Homogeneous System of Linear Equations Over Supertropical Algebra

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Abstract – In this paper we discuss the characterization of the solution system $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$ over supertropical algebra. The Cramer method is used to solve system of linear equations. We show that the $n \times n$ system of linear equations over supertropical algebra has a unique solution, or an infinite number of solutions.

1. INTRODUCTION

Tropical algebra is idempotent semirings and semifields. Max-plus algebra is one of many idempotent semirings. Max-plus algebra is defined as $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, where $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$ with $\mathbb{R}$ is the set of real numbers, $\varepsilon \stackrel{\text{def}}{=} -\infty$, $a \oplus b \stackrel{\text{def}}{=} \max\{a,b\}$and $a \otimes b \stackrel{\text{def}}{=} a + b$ for every $a, b \in \mathbb{R}_\varepsilon$. We can show that $(\mathbb{R}_\varepsilon, \oplus, \otimes)$is an idempotent semiring, where for every $a \in \mathbb{R}_\varepsilon$, $a \oplus a = a$ holds. Moreover $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is a semifield, where for every $a \in \mathbb{R}$ there exist $-a$ such that $a \otimes (-a) = 0$ [1]. In contrast to conventional linear algebra, there are no inverse elements with respect to $\oplus$ in $\mathbb{R}_{\max}$. It also causes difficulty when solving linear systems of equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$. Consider the following equation $x_1 \oplus 12 = 2 \Leftrightarrow \max\{x_1, 12\} = 2$. Clearly, this equation has no solutions in $\mathbb{R}_{max}$. Therefore a new structure that generalizes max-plus algebra is constructed and it is called extended tropical semiring, denoted as $T = \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^v$. That is a semiring whose neutral element $\varepsilon \stackrel{\text{def}}{=} -\infty$ and unity element $e \stackrel{\text{def}}{=} 0$. $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^v = \{a^v : a \in \mathbb{R}\}$. $\mathbb{R}^v_{-\infty} = \mathbb{R}^v \cup \{-\infty\}$ is called ideal of T, and $v : T \to \mathbb{R}^v_{-\infty}$ is called the ghost map satisfying $v(a) = a, \forall a \in \mathbb{R}^v_{-\infty}$ and $v(a) = a^v, \forall a \in \mathbb{R}$. Generally, the extension of tropical algebra is called supertropical algebra. This paper concentrates on characteristic of the solutions of non homogeneous system of linear equations $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$ over supertropical algebra.

2. METHODS

In this section we explain the basic concept and notation of max-plus algebra, tropical algebra, and supertropical algebra. For further details see [2] and [3].

2.1 Max-Plus Algebra

Definition 1. [1] Max-plus algebra is defined as $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, where $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$ with $\mathbb{R}$ is the set of real numbers, $\varepsilon \stackrel{\text{def}}{=} -\infty$, $a \oplus b \stackrel{\text{def}}{=} \max\{a,b\}$and $a \otimes b \stackrel{\text{def}}{=} a + b$ for every $a, b \in \mathbb{R}_\varepsilon$.

We can show that $(\mathbb{R}_\varepsilon, \oplus, \otimes)$is an idempotent semiring, where for every $a \in \mathbb{R}_\varepsilon$, $a \oplus a = a$ holds. Moreover $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is a semifield, where for every $a \in \mathbb{R}$ there exist $-a$ such that $a \otimes (-a) = 0$.

2.2 Tropical Algebra

Tropical algebra is idempotent semirings and semifields. Based on Definition 1, max-plus algebra is idempotent semirings and semifields, so it is a subclass of tropical algebra.

2.3 Extended Tropical Semiring

Definition 2. [2] Extended tropical semiring $T = \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^v$ is a semiring whose neutral element $\varepsilon \stackrel{\text{def}}{=} -\infty$ and unity element $e \stackrel{\text{def}}{=} 0$. $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^v = \{a^v : a \in \mathbb{R}\}$. $\mathbb{R}^v_{-\infty} = \mathbb{R}^v \cup \{-\infty\}$ is called ideal of T, and $v : T \to \mathbb{R}^v_{-\infty}$ is called the ghost map satisfying $v(a) = a, \ \forall a \in \mathbb{R}^v_{-\infty}$ and $v(a) = a^v, \forall a \in \mathbb{R}$.

We use the notation $a, b \in \mathbb{R}$ for reals, $a^v, b^v \in \mathbb{R}^v$where $a, b \in \mathbb{R}$, and $x, y \in T$.

Definition 3. [2] The Partial order relation $\prec$ on T is defined as

1. $-\infty \prec x, \forall x \in T \setminus \{-\infty\}$.
2. For every real numbers $a < b$ then $a \prec b, a \prec b^v, a^v \prec b,$ and $a^v \prec b^v$.

3. $a \prec a^{v}$ for every $a \in \mathbb{R}$.

Axiom 1. [2]The axiom of the extended tropical semiring T are

1. $-\infty \oplus x = x \oplus -\infty = x$ for every $\in T$.
2. $x \oplus y = \max_{\prec}\{x, y\}$ unless $x = y$.
3. $a \oplus a = a^{v} \oplus a^{v} = a \oplus a^{v} = a^{v} \oplus a = a^{v}$.
4. $-\infty \otimes x = x \otimes -\infty = -\infty$ for every $x \in T$.
5. $a \otimes b = a + b$ for every $a, b \in \mathbb{R}$.
6. $a^{v} \otimes b = a \otimes b^{v} = a^{v} \otimes b^{v} = (a + b)^{v}$.

2.4 Supertropical Algebra

Definition 4. [2] Semiring with ghost is a triplet $(R,\ \mathcal{G}_0,\ v)$ where R is a semiring with neutral element 0_R and unity element 1_R, $\mathcal{G}_0 = \mathcal{G} \cup \{0_R\}$ is called the ghost ideal, and $v:\ R \to \mathcal{G}_0$ is called the ghost map satisfying

$$a \oplus a = v(a), \forall a \in R.$$

$\mathcal{T} = R \setminus \mathcal{G}_0$ is the set of tangible elements, and $\mathcal{G}$ is the set of ghost elements.

Definition 5. [2] A supertropical semiring is a semiring with ghost $(R,\ \mathcal{G}_0,\ v)$that has the additional properties, $\forall a, b \in R$ such that if $a^{v} = b^{v}$ then $a \oplus b = a^{v}$ and if $a \neq b$ then $a \oplus b \in \{a, b\}$.

Definition 6. [3] The ghost surpassing relation $\vDash$ on R is defined as

$$a \vDash b \text{ if } a = b \oplus c, \text{ for some } c \in \mathcal{G}_0.$$

2.5 Matrices over Supertropical Algebra

The square matrices of size $n \times n$ in supertropical algebra are denoted by $M_n(R)$with entries in R. For every $n \in \mathbb{N}$ with $n \neq 0$, operation $\oplus$ and $\otimes$ of R can be extended to matrices $M_n(R)$.

We collect some basic definitions about matrices over supertropical algebra.

Definition 7. [3] Determinant of matrices $A \in M_n(R)$ is defined as
$|A| = \oplus_{\pi \in S_n} a_{1,\pi(1)} \otimes a_{2,\pi(2)} \otimes \ldots \otimes a_{n,\pi(n)}$, where S_n is a grup permutation.

Definition 8. [3] Let $A \in M_n(R)$, minor $M_{i,j}$is defined as determinant obtained by deleting the i row and j column of A. The adjoint matrices of A denoted by $\text{adj}(A) = (\text{Cof}(A))^{T}$ where $\text{Cof}(A) = \begin{bmatrix} \text{cof}_{11} & \cdots & \text{cof}_{1n} \\ \vdots & \ddots & \vdots \\ \text{cof}_{n1} & \cdots & \text{cof}_{nn} \end{bmatrix}$ and $cof_{i,j} = M_{i,j}$.

Definition 9. [4] A matrices $A \in M_n(R)$ is called non singular if $|A| \in \mathcal{T}$ and singular if $|A| \in \mathcal{G}_0$.

Definition 10. A pseudo-identity matrices of size $n \times n$ denoted by $I_{\mathcal{G}}$, i.e. the main diagonal elements of the matrices are equal to e and the other elements are equal to ε or $a^{v} \in \mathcal{G}$.

Definition 11. [4] Suppose $A \in M_n(R)$, pseudo-invers A^{∇} of A is defined as $A^{\nabla} = \frac{1_R}{|A|} \otimes \text{adj(A)}$, if $|A| \in \mathcal{T}$.

3. RESULTS AND DISCUSSION

In this section we discuss the characteristic of the solutions of $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$ over supertropical algebra. We focus on extended tropical semiring, that is $T = \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^{v}$ with $\mathcal{T} = \mathbb{R}$, $\mathcal{G} = \mathbb{R}^{v}$ and $R = T$. System of linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ will be weakened $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$. Therefore, we introduce a ghost surpass relation on R.

Definition 12. [3] Let $a, b \in R$ then $a \vDash b \Leftrightarrow a = b \oplus c$, for some $c \in \mathcal{G}_0$.
Definition 13. [3] Let $a \in R, b \in \mathcal{T}$ then $a \vDash b \Leftrightarrow a \oplus b \in \mathcal{G}_0$.
Definition 14. [3] Let $a, b \in \mathcal{T}$ then $a \vDash b \Leftrightarrow a = b$.
Definition 15. [3] Let $a \in \mathcal{G}_0,\ b \in R$ then $a \vDash b \Leftrightarrow a \succcurlyeq_{v} b$.

Now we give the solution set of some basic ghost surpass on R. Suppose $a \in \mathcal{T}$ and $x \in R$.
1. $x \vDash a$. The solution set of x is given by $\{a\} \cup \{b^{v} | b \in \mathcal{T} \text{ and } b \succcurlyeq_{v} a\}$.
2. $x \vDash a^{v}$. The solution set of x is given by $\{b^{v} | b \in \mathcal{T} \text{ and } b \succcurlyeq_{v} a\}$.
3. $x \vDash \varepsilon$. The solution set of x is given by $\{\varepsilon\} \cup \{b^{v} | b \in \mathcal{T}\}$.

Lemma 1. Let $a \in \mathcal{T}, b \in R$. For every $x \in R$ then $a \otimes x \vDash b \Leftrightarrow x \vDash a^{\otimes -1} \otimes b$.

Theorem 1. Let $a \in \mathcal{T}, b \in \mathcal{T}$. For every $x \in \mathcal{T}$ then $x = a^{\otimes -1} \otimes b$ is the unique solution of equation

$a \otimes x \vDash b$.
Proof : We will prove that $x = a^{\otimes -1} \otimes b$ is the unique solution of $a \otimes x \vDash b$.
Suppose $y \in \mathcal{T}$ is another solution of $a \otimes x \vDash b$ then $a \otimes y \vDash b \Leftrightarrow y \vDash a^{\otimes -1} \otimes b$. It is known that $x \vDash a^{\otimes -1} \otimes b$ is the solution of $a \otimes x \vDash b$ then $\begin{cases} x \vDash a^{\otimes -1} \otimes b \\ y \vDash a^{\otimes -1} \otimes b \end{cases}$. Since $x \in \mathcal{T}$ and $y \in$, based on the Definition 14 we obtain $x \vDash y \Leftrightarrow x = y$. So it is clear that if $y \in \mathcal{T}$ is another solution of $a \otimes x \vDash b$ then $y = x = a^{\otimes -1} \otimes b$. We get the solution of $a \otimes x \vDash b$ is the unique solution $x = a^{\otimes -1} \otimes b$.

Theorem 2. Let $a \in \mathcal{T}$ and $b \in \mathcal{T}$. All the solution of $a \otimes x \vDash b$ can be written as

$$x_t = x \oplus t^{v} \text{ for some } t^{v} \in \mathcal{G}_0.$$

Proof : Based on Definition 13 we obtain $a \otimes x \vDash b \Leftrightarrow a \otimes x \oplus b \in \mathcal{G}_0$.
Based on Theorem 1, we know that $x = a^{\otimes -1} \otimes b$ is the unique solution of $a \otimes x \vDash b$. For every $x \in \mathcal{T}$, it is easy to see that $a \otimes t^{v} \vDash \varepsilon$ since $a \otimes t^{v} \in \mathcal{G}_0$.
By adding the equation $a \otimes t^{v} \vDash \varepsilon$ to $a \otimes x \oplus b \vDash \varepsilon$ then
$\Leftrightarrow (a \otimes t^{v}) \oplus (a \otimes x) \oplus b \vDash \varepsilon$
$\Leftrightarrow a \otimes (x \oplus t^{v}) \vDash b$
By left multiplying $a \otimes (x \oplus t^{v}) \vDash b$ by $a^{\otimes -1}$ we obtain
$\Leftrightarrow a^{\otimes -1} \otimes a \otimes (x \oplus t^{v}) \vDash a^{\otimes -1} \otimes b$
$\Leftrightarrow x \oplus t^{v} \vDash a^{\otimes -1} \otimes b$
Assume $x_t = x \oplus t^{v}$ then $x_t \vDash x$. Based on Definition 12, we can write
$x_t \vDash x \Leftrightarrow x_t = x \oplus t^{v}$ for some $t^{v} \in \mathcal{G}_0$. If $t \geqslant_v x$ then $x_t = t^{v}$.
Therefore, all the solution of $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$ can be written as $x_t = x \oplus t^{v}$ for some $t^{v} \in \mathcal{G}_0$.

Moreover, ghost surpass relation on Rcan be extended to the vector case.
Definition 16. [3] Suppose $\boldsymbol{u} \in R^n$ and $\boldsymbol{w} \in \mathcal{T}_0{}^{n}$ then $\boldsymbol{u} \vDash \boldsymbol{w} \Leftrightarrow \boldsymbol{u} \oplus \boldsymbol{w} \in \mathcal{G}_0{}^{(n)}$.

Definition 17. [3] Let $A \in M_n(R)$, $\boldsymbol{x} \in R^n$ and $\boldsymbol{b} \in \mathcal{T}_0{}^{n}$ then the system of

$$A \otimes \boldsymbol{x} \vDash \boldsymbol{b} \text{ is equivalent with } A \otimes \boldsymbol{x} \oplus \boldsymbol{b} \in \mathcal{G}_0{}^{(n)}.$$

Next, we will discuss about the solution system of linear equations $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$ over supertropical algebra by using Cramer formula.
Lemma 2. [3] Let $A \in M_n(R)$ then $A \otimes \mathrm{adj(A)} \vDash |A| \otimes I_A$.
Proof : We will prove that $A \otimes \mathrm{adj(A)} \vDash |A| \otimes I_A$. Based on Definition 13, to prove $A \otimes \mathrm{adj(A)} \vDash |A| \otimes I_A$, it is same by proving that $\left(A \otimes \mathrm{adj(A)}\right) \oplus (|A| \otimes I_A) \in \mathcal{G}_0{}^{(n)}$.
$\left(A \otimes \mathrm{adj(A)}\right) \oplus (|A| \otimes I_A) = [A_{i,j}] \otimes [Cof_{j,i}(A)] \oplus |A| \otimes I_A = (|A| \otimes I_A) \oplus (|A| \otimes I_A) \in \mathcal{G}_0{}^{(n)}$,
thus, it can be concluded that $A \otimes \mathrm{adj(A)} = |A| \otimes I_A$.

Lemma 3. Let $A \in M_n(R)$ the matrices$A \in M_n(R)$ are partitioned into the following for $A = \begin{bmatrix} F & G \\ H & a_{n,n} \end{bmatrix}$ where F is $(n-1) \times (n-1)$ matrices of A, H is $(n-1) \times 1$ matrices of A, G is $1 \times (n-1$ matrices of A, $a_{n,n}$is tangible element of A. Then $|A| = \begin{vmatrix} F & G \\ H & a_{n,n} \end{vmatrix} = \left(|F| \otimes a_{n,n}\right) \oplus (H \otimes \mathrm{adj}(F) \otimes G)$.

The following is Cramer formula in supertropical algebra.
Theorem 3. [3] Let $A \in M_n(R)$ and $\boldsymbol{b} \in \mathcal{T}_0{}^{n}$. Then every solution $\boldsymbol{x} \in R^n$ of the linear system

$$A \otimes \boldsymbol{x} \vDash \boldsymbol{b} \tag{1}$$

satisfies

$$|A| \otimes \boldsymbol{x} \vDash \mathrm{adj(A)} \otimes \boldsymbol{b} \tag{2}$$

then the solution $\boldsymbol{x} = |A|^{\otimes -1} \otimes (\mathrm{adj(A)} \otimes \boldsymbol{b})$.

Proof : We will prove that $\boldsymbol{x} \in R^n$ satisfies $|A| \otimes \boldsymbol{x} \vDash \mathrm{adj(A)} \otimes \boldsymbol{b}$.
Based on Lemma 2 we obtain $A \otimes \mathrm{adj(A)} \vDash |A| \otimes I_A$.
Assumed $|A| \in \mathcal{T}$, by right multiplying of the $A \otimes \mathrm{adj(A)} \vDash |A| \otimes I_A$ by $|A|^{-1} \otimes \boldsymbol{b}$, we obtain
$\left(A \otimes \mathrm{adj(A)}\right) \otimes \left(|A|^{\otimes -1} \otimes \boldsymbol{b}\right) \vDash (|A| \otimes I_A) \otimes \left(|A|^{\otimes -1} \otimes \boldsymbol{b}\right)$
$\Leftrightarrow A \otimes \left(\mathrm{adj(A)} \otimes |A|^{\otimes -1} \otimes \boldsymbol{b}\right) \vDash \left(|A| \otimes |A|^{\otimes -1}\right) \otimes (I_A \otimes \boldsymbol{b})$
$\Leftrightarrow A \otimes \left(|A|^{\otimes -1} \otimes \mathrm{adj(A)} \otimes \boldsymbol{b}\right) \vDash (I_A \otimes \boldsymbol{b})$
$\Leftrightarrow A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$
we see that $\boldsymbol{x} = |A|^{\otimes -1} \otimes \mathrm{adj(A)} \otimes \boldsymbol{b}$ is the solution of $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$.

Theorem 4. [3] Let $A \in M_n(R)$ and $\boldsymbol{b} \in \mathcal{T}_0{}^n$. If the vector $(\mathrm{adj(A)} \otimes \boldsymbol{b}) \in \mathcal{T}_0{}^n$ and $|A| \in \mathcal{T}$ then $\boldsymbol{x} = |A|^{\otimes -1} \otimes (\mathrm{adj(A)} \otimes \boldsymbol{b})$ is the unique solution of $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$.

Proof: We will prove that $\boldsymbol{x} = |A|^{\otimes -1} \otimes (\mathrm{adj(A)} \otimes \boldsymbol{b})$ is the unique solution of $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$. We will proof it by induction on the dimension n (size of the matrices A).
When $n = 1$ the result of obvious. Based on Theorem 1, we obtain $a \otimes x \vDash b$ then $x = a^{\otimes -1} \otimes b$.
Assume $n = k - 1$ is true,
we will prove that $n = k$ also holds.
Now, the matrices $A \in M_n(R)$, $\boldsymbol{b} \in \mathcal{T}_0{}^n$ and $\boldsymbol{x} \in \mathcal{T}_0{}^n$ are partitioned into the following form
$A = \begin{bmatrix} H & a_{1,n} \\ F & G \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ B \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} X \\ x_n \end{bmatrix}$ where$H$ is $1 \times (n-1$ matrices of A, $F$is $(n-1) \times (n-1)$ matrices of A, G is $(n-1) \times 1$ matrices of A, $B$is $(n-1) \times 1$ matrices of $\boldsymbol{b}$, X is $(n-1) \times 1$ matrices of $\boldsymbol{x}$, $a_{1,n}$ is tangible element of A, b_1 is tangible element of $\boldsymbol{b}$, and x_n is tangible element of $\boldsymbol{x}$.
so that the equation $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$ can be written as $\begin{bmatrix} H & a_{1,n} \\ F & G \end{bmatrix} \otimes \begin{bmatrix} X \\ x_n \end{bmatrix} \vDash \begin{bmatrix} b_1 \\ B \end{bmatrix}$

$$A \otimes \boldsymbol{x} \vDash \boldsymbol{b} \Leftrightarrow \begin{cases} (H \otimes X) \oplus (a_{1,n} \otimes x_n) \vDash b_1 & (3) \\ (F \otimes X) \oplus (G \otimes x_n) \vDash B & (4) \end{cases}$$

by left multiplying the equation (4) by F^{∇}we obtain $F^{\nabla} \otimes (F \otimes X) \vDash F^{\nabla} \otimes (B \oplus (G \otimes x_n))$
$\Leftrightarrow X \vDash (F^{\nabla} \otimes B) \oplus (F^{\nabla} \otimes G \otimes x_n)$
$\Leftrightarrow X \vDash F^{\nabla} \otimes (B \oplus (G \otimes x_n))$
$\Leftrightarrow X \vDash \left(|F|^{\otimes -1} \otimes \mathrm{adj(F)}\right) \otimes (B \oplus (G \otimes x_n))$
(5)
using the substitution, we replace equation (5) in (3) we obtain $(H \otimes X) \oplus (a_{1,n} \otimes x_n) \vDash b_1$
$\Leftrightarrow H \otimes \left(|F|^{\otimes -1} \otimes \mathrm{adj(F)}\right) \otimes (B \oplus (G \otimes x_n)) \oplus (a_{1,n} \otimes x_n) \vDash b_1$
$\Leftrightarrow \left(|F|^{\otimes -1} \otimes H \otimes \mathrm{adj(F)} \otimes B\right) \oplus \left(|F|^{\otimes -1} \otimes H \otimes \mathrm{adj(F)} \otimes (G \otimes x_n)\right) \oplus (a_{1,n} \otimes x_n) \vDash b_1$
(6)
by left multiplying the equation (6) by $|F|$, we obtain
$\Leftrightarrow (H \otimes \mathrm{adj(F)} \otimes B) \oplus (H \otimes \mathrm{adj(F)} \otimes (G \otimes x_n)) \oplus |F| \otimes (a_{1,n} \otimes x_n) \vDash |F| \otimes b_1$
$\Leftrightarrow |F| \otimes (a_{1,n} \otimes x_n) \oplus (H \otimes \mathrm{adj(F)} \otimes (G \otimes x_n)) \vDash (|F| \otimes b_1) \oplus (H \otimes \mathrm{adj(F)} \otimes B)$
$\Leftrightarrow x_n \otimes \left((|F| \otimes a_{1,n}) \oplus (H \otimes \mathrm{adj(F)} \otimes G)\right) \vDash (|F| \otimes b_1) \oplus (H \otimes \mathrm{adj(F)} \otimes B)$
$\Leftrightarrow x_n \vDash \dfrac{(|F| \otimes b_1) \oplus (H \otimes \mathrm{adj(F)} \otimes B)}{\left((|F| \otimes a_{1,n}) \oplus (H \otimes \mathrm{adj(F)} \otimes G)\right)}$
based on Lemma 3, we obtain $|A| = \left((|F| \otimes a_{1,n}) \oplus (H \otimes \mathrm{adj(F)} \otimes G)\right)$, thus

$$(\mathrm{adj(A)} \otimes b)_n = (|F| \otimes b_1) \oplus (H \otimes \mathrm{adj(F)} \otimes B)$$

Finally we obtain $x_n \vDash \frac{(\mathrm{adj(A)} \otimes b)_n}{|A|} \Leftrightarrow x_n \vDash |A|^{\otimes -1} \otimes (\mathrm{adj(A)} \otimes b)_n$.
Since $(\mathrm{adj(A)} \otimes \boldsymbol{b}) \in \mathcal{T}_0{}^n$ and $|A| \in \mathcal{T}$ then $x_n \in \mathcal{T}_0{}^n$ for any n.
So it is clear that $\boldsymbol{x} = |A|^{\otimes \boldsymbol{-1}} \otimes (\mathrm{adj(A)} \otimes \boldsymbol{b})$ is the unique solution of $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$.

Let D_{xi} be the determinant of the matrices obtained by replacing the $i - th$ column of A with $\boldsymbol{b}$, then $(\mathrm{adj(A)} \otimes b)_i = D_{xi}$. Assume $D = |A|$ then the equation (2) is equivalent to
$\Leftrightarrow D \otimes \boldsymbol{x} \vDash \mathrm{adj(A)} \otimes \boldsymbol{b}$
$\Leftrightarrow \boldsymbol{x} \vDash \mathrm{adj(A)} \otimes \boldsymbol{b} \otimes D^{\otimes -1}$
$\Leftrightarrow x_i \vDash (\mathrm{adj(A)} \otimes \boldsymbol{b})_i \otimes D^{\otimes -1}$
$\Leftrightarrow x_i \vDash D_{xi} \otimes D^{\otimes -1}$
If $D_{xi} \in \mathcal{T}$ and $D \in \mathcal{T}$ we obtain $x_i \in \mathcal{T}_0{}^n$ for any $i \in n$. Then $x_i = D_{xi} \otimes D^{\otimes -1}$.
We see that it is exactly the Cramer formula in classical algebra.

4. CONCLUSION

Characterization of the solution of $n \times n$ system of non homogeneous of linear equations $A \otimes \boldsymbol{x} \vDash \boldsymbol{b}$over supertropical algebra has a unique solution if and only if $|A| \in \mathcal{T}$ and $(\mathrm{adj(A)} \otimes \boldsymbol{b}) \in \mathcal{T}_0{}^n$, and has an infinite number of solutions if and only if $|A| \in \mathcal{G}_0$.

5. REFERENCES

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