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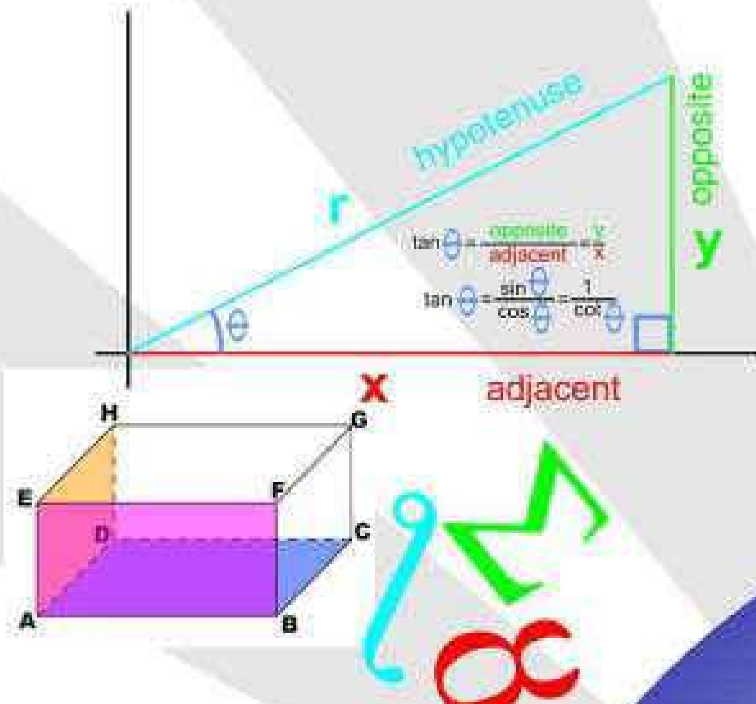

June 6 - 8, 2011

Hosted by :

Departement of Mathematics

Faculty of Mathematics and Natural Sciences

The International Conference on Numerical Analysis and Optimization (ICeMATH2011)

The field of numerical analysis predates the invention of modern computers by many centuries. Numerical analysis is the area of mathematics and computer science that creates, analyzes, and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real-world applications of algebra, geometry, and calculus, and they involve variables which vary continuously.

On the other hand, Numerical Optimization is defined as a scientific approach in finding the finest solution of a particular problem that is interpreted in mathematical models. Hence, the combination of numerical analysis with numerical optimization is highly important for scientific efforts in the areas of developmental work as well as humanity in general.

Therefore, on the occasion of the 50th anniversary of its founding celebration, Universitas Ahmad Dahlan (UAD) with the collaboration of Journal KALAM has initiated **The International Conference on Numerical Analysis and Optimization (ICeMATH 2011)** to be held at Yogyakarta, Indonesia.

Objectives:

- Provide a platform for researchers, professionals, and academicians to exchange ideas and discuss their research findings.
- Encourage future collaborations between participants.
- Provide room for researchers to discuss their thoughts and views on the development of this field that can contribute towards future works as well as being a very beneficial program for all participants.

Topic of Discussions:

Numerical Analysis, Numerical Methods, Operations Research, Mathematics, Statistics, Numerical Optimization, Differential Equation, Applied Mathematics and Statistics, Interval Mathematics, Fuzzy, Computational Mathematics, Combinatory, Algebra, Engineering Mathematics, Mathematics Education

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6^{TH} – 8^{TH} JUNE 2011

UNIVERSITAS AHMAD DAHLAN, YOGYAKARTA, INDONESIA

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Part VIII - Statistics

WEAKLY REACHABILITY AND WEAKLY OBSERVABILITY OF LINEAR SYSTEM OVER MAX PLUS ALGEBRA

Tri Siwi Nasrulyati[1], Subiono[2], Erna Apriliani[3]

Abstract. This paper discusses about the properties of linear system in max plus algebra. These properties are weakly reachability and weakly observability. In this case, the asticity of the system plays big role in these properties as the necessary and sufficient conditions. Furthermore, we will also discuss the duality of those properties. Finally, to make the discuss simple, we will gift the example.

Keywords and Phrases: *Max plus linear system, reachability, observability..*

1. INTRODUCTION

The systems are changed accordingly to changes of time. But there are also system which are changes accordingly to changes of event. Thos a kind of systems are known as event driven systems. Max plus algebra is a method which can formulate the driven event systems. These systems will be linear over max plus algebra [6].

The study of max plus algebra and its linear systems are developed widely; this study is including the theory of weakly reachability and weakly observability of the systems. The weakly reachability means by a control system from any initial state to any other state. The systems are controlled by using the input. The difference between reachability and controllability is depending on the initial state. The reachability , is the controlling the system from any initial state to any other state. But controllability is the controlling from the origin state to any other state. The concept of the reachability in the max plus algebra is not too different from the definitions of the controllability in continues system and the concept about the observability in max plus algebra is also different from the observability definitions in continues systems [3].

In this paper, we will discuss about the theory of weakly reachability and weakly observability in the linear max plus systems. In the discussion we will use the definition of reachable and observable set. Furthermore, we also discuss about the duality among these properties and give them example.

1.1 Max Plus Algebra

In the section we explain the basic concept and notation. There are a lot of references which explain about max plus algebra, the detail information can be found in [2] and [7]. In the max plus algebra, for any $a, b \in \mathrm{R}_{\max} = \{-\infty\} \cup \{\mathrm{R}\}$ defined two operations, $\oplus$ and $\otimes$ as follows

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \otimes b = a + b$$

Definition 1. For all $x, y, z \in \mathrm{R}_{\max}$ satisfies: 1) Associative concerning $\otimes$ and $\oplus$. 2) Commutative concerning $\otimes$ and $\oplus$. 3) Distributive. 4) Zero element of $\oplus$. 5) Unit element of $\otimes$. 6) Multiplicative invert if $x \neq \varepsilon$ then there is y such that $x \otimes y = e$ and y is the one and only. 7) Absorption element of $\otimes$. 8) Idempotent in addition.

Definition 2. For $x \in \mathrm{R}_{\max}$ and $n \in \mathrm{N}$ satisfies $x^{\otimes n} = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n\ times}$

Power in max plus algebra can be derived as multiplication in conventional algebra $x^{\otimes n} = nx$, such that in generally satisfies as follows:
(i) If $x \neq \varepsilon$, then $x^{\otimes 0} = e = 0$ (ii) if $\alpha \in \mathrm{R}$, then $x^{\otimes \alpha} = \alpha \otimes x$ (iii) if $k > 0$ then $\varepsilon^{\otimes k} = \varepsilon$, $\varepsilon^{\otimes k}$ is undefined for $k \leq 0$,

1.2 Matrix Over Max Plus Algebra

The set of matrices size $n \times m$ in max plus algebra denoted by $\mathrm{R}_{\max}^{n \times m}$ with $n, m \in \mathrm{N}$ and n or $m \neq 0$. Element $A \in \mathrm{R}_{\max}^{n \times m}$ *i-th* row *j-th* column denoted by $a_{i.j}$ or $[A]_{i.j}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. Matrix A can be written as

$$A = \begin{pmatrix} a_{1.1} & a_{1.2} & \cdots & a_{1.m} \\ a_{2.1} & a_{2.2} & \cdots & a_{2.m} \\ \vdots & \vdots & \ddots & \\ a_{n.1} & a_{n.21} & \cdots & a_{n.m} \end{pmatrix}$$

In max plus algebra operation + and $\times$ from vector and matrix are replaced with $\oplus$ and $\otimes$ $\oplus$ from vector and matrices are replaced with $\oplus$ and $\otimes$.

Definition 3.

1) For any $A, B \in \mathrm{R}_{\max}^{n \times m}$ and $a \in \mathrm{R}$ define an addition operation $A \oplus B$ as $[A \oplus B]_{i.j} = a_{i.j} \oplus b_{i.j} = \max(a_{i.j}, b_{i.j})$
2) For $A, \in \mathrm{R}_{\max}^{n \times p}$ and $B \in \mathrm{R}_{\max}^{p \times m}$ then we define operation $A \otimes B$ as
$$[A \otimes B]_{i,j} = \bigoplus_{k=1}^{p} \left(a_{i,k} \otimes b_{k,j} \right) = \max_{k \in p} \left\{ a_{i,k} \otimes b_{k,j} \right\} = \max_{k \in p} \left\{ a_{i,k} + b_{k,j} \right\}$$
3) The transpose of matrix A denoted by A^T and defined as usual we find in conventional algebra by $[A^T]_{i.j} = [A]_{j.i}$.
4) Identity matrix of size $n \times n$ in max plus is denoted by E_n and define as
$$[E]_{i.j} = \begin{cases} e & \text{jika } i = j \\ \varepsilon & \text{jika } i \neq j \end{cases}$$
5) For square matrix and $k \in \mathrm{N}$, *k-th* power of A denoted by $A^{\otimes k}$ and defined as $A^{\otimes k} = \underbrace{A \otimes A \otimes A \ldots \otimes A}_{k\ kali}$, for $k = 0, A^{\otimes 0} = E_n$.
6) For matrix $A \in \mathrm{R}_{\max}^{n \times m}$ and scalar $\alpha \in \mathrm{R}_{\max}$, $\alpha \otimes A$ define by
$[\alpha \otimes \mathrm{A}]_{\mathrm{i.j}} = \alpha \otimes [A]_{i.j}$ For $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

1.3 Linear Max plus System

Let the discrete event system be the event driven systems with a discrete state (as in production system, storage with finite capacity system, railway system, logistic system and so on). This state is described by the equation below:

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \quad (1)$$

$$y(k) = C \otimes x(k) \quad (2)$$

With $A \in \mathrm{R}^{n\times m}_{\max}, B \in \mathrm{R}^{m\times n}_{\max}, C \in \mathrm{R}^{p\times n}_{\max}$ and x represents the state, u represents the input and y represents the output, k is the event index which are $k = 0, 1, 2 \ldots$. Both equation (1) and (2) are called by linear max plus system [6].

2. REACHABILITY

In this section discuss the discrete event systems which are formulated in to max plus algebra, so we get the linear one. This discussion will be done in the systems that many reaches a final condition with all of it component are greater than the final one without any input. This kinds of system later known by weakly reachable system. Using (1) in recursive fashion, the state system can be written to each event index $k = 1, 2 \ldots$ q, as follows:

For $k=0$ then $X(1) = A \otimes X(0) \oplus B \otimes U(1)$

For $k=1$ then $X(2) = A \otimes X(1) \oplus B \otimes U(2) = A^2 X(0) \oplus ABU(1) \oplus BU(2)$

For $k=2$ then $X(3) = A \otimes X(2) \oplus B \otimes U(3) = A^3 X(0) \oplus A^2 BU(1) \oplus ABU(2) \oplus BU(3)$

so, to q-step event we get:

$$X(q) = A^q \otimes X(0) \oplus \left[B\ AB\ A^2B\ \cdots A^{q-1}B\right] \otimes \left[U(q)\ U(q-1)\ \ U(q-2)\ U(Q-3)\ \cdots U(1)\right]^T \quad (3)$$

From (3) we obtain the reachability matrix notated by $\Gamma_q = [B\ \ AB\ \ A^2B\ \cdots\ A^{q-1}B]$. This matrix is the one which influence the reachability of the system, the input series defined by $U_q = [U_q\ U_{q-1}\ \ \cdots U_1]^T$, so the state of q-step event can be written by:

$$X(q) = A^q \otimes X(0) \oplus \Gamma_q \otimes U_q \quad (4)$$

Definition 4. Reachable State. Given $X(0) \in \mathrm{R}^n_{\max}$, a state $X \in \mathrm{R}^n$ is reachable in q-step from $X(0)$ if there exists a control sequence $\{U(1), U(2), \cdots, U(q)\} \in \mathrm{R}_{\max}$, which achieves $X = X(q)$.

Definition 5. Reachable Set. Let $X(0) \in \mathrm{R}^n_{\max}$, be the initial condition, the set of all of the state $X \in \mathrm{R}^n$ that can be reached at q-step event (with q should be positive integer) is defined as follows:

$$\Omega_{q.X(0)} = \{X \in \mathrm{R}^n : X = A^q \otimes X(0) \oplus \Gamma_q \otimes U_q,\ \text{where}\ \ U_q \in \mathrm{R}^{p\times q}_{\max}\}$$

Theorem 1. Given an initial state $X(0) \in \mathrm{R}^n_{\max}$ and a state $X \in \Omega_{q.X(0)}$ if and only if

$$X = \Gamma_q \otimes (-\Gamma_q^T \otimes' X) \oplus A^q \otimes X(0) \quad (5)$$

In which case $-\Gamma_q^T \otimes' X = U_q$ is a controller drives state from $X(0)$ to $X = X(q)$.

Proof. If $X \in \Omega_{q.X(0)}$, then according Definition 1, there is U_q such that the q-step state $X = A^q \otimes X(0) \oplus \Gamma_q \otimes U_q$, is reached. Because of that $\Gamma_q \otimes U_q \le X$. From [2] and [7], we get $U_q = -\Gamma_q^T \otimes' X$ is the biggest solution, then $\Gamma_q \otimes (-\Gamma_q^T \otimes' X) \le X$. So we get

$$\Gamma_q \otimes U_q \le \Gamma_q \otimes (-\Gamma_q^T \otimes' X) \le X \quad (6)$$

With adding $A^q \otimes X(0)$ to each term in (6), we obtain:

$$A^q \otimes X(0) \oplus \Gamma_q \otimes U_q \leq A^q \otimes X(0) \oplus \Gamma_q \otimes (-\Gamma_q^T \otimes' X) \leq A^q \otimes X(0) \oplus X$$

Then we can write that $\Gamma_q \otimes (-\Gamma_q^T \otimes' X) \oplus A^q \otimes X(0) = X$, so equation (5) satisfied.

In max plus case, different from the continuo one, because the maximum operation, $A^q \otimes X(0) \oplus \Gamma_q \otimes U_q$ could not be equal to the states which are less than $A^q \otimes X(0)$. In this paper, we focus the analyzing at the systems which reach a state with all of the components that greater than the final state. The condition of the system is called weakly reachable system.

Definition 6. Q-step Weakly Reachable [3]. A system is said to be q-step weakly reachable, if given any *X(0)*, a controller sequence exist such that each component of the terminal state *X(q)* can be made greater than the unforced terminal state $A^q \otimes X(0)$, there exist U_q such that $(X(q))_j > (A^q \otimes X(0))_j$ for $j = 1,2,\cdots,n$.

Before we discuss more about the weakly reachability, we will give the definition as acticity first.

Definition 7. Asticity [3]. A $n \times m$ $G = \{g_{ij}\}$, is termed row astic if for each row $i = 1,2,\cdots,n$, $\oplus_{j=1}^{m} g_{ij} \in R.$ Matrix G is termed column astic if for each column $j = 1,2,\cdots,m$ the $\oplus_{i=1}^{m} g_{ji} \in R.$ A matrix is termed doubly astic if it in both row and column astic.

This asticity property is necessary and sufficient condition for the system to be called as weakly reachability or weakly observability.

Theorem 2. [3] A system is q-step weakly reachable if and if Γ_q is row astic.

Proof. If Γ_q is row astic, with a great enough U_q , $(\Gamma_q \otimes U_q)_j > (A^q \otimes X(0))_j$, for $j = 1,2,\cdots,n$. From the Definition 6 if a system q-step weakly reachable, then $(\Gamma_q \otimes U_q)_j > (A^q \otimes X(0))_j$ should be satisfied. So $(\Gamma_q \otimes U_q)_j$ should be finite for each *j*, because of that Γ_q has to be row astic. Then the system is q-step weakly reachable.

Actually, row astic condition for the reachability matrix Γ_q is needed to find that there is as least an input for each state internal transition systems. Cayley-Hamilton theorem in max plus can be used to show that if a system is not weakly reachable at q-step, then the system is also not weakly reachable at step which are more than q.

3. OBSERVABILITY

A system is observable if there is a final state of the system that can to determine from the measurement of the output. Because the inverse concerning the addition operator is not existing, cause the observability of the system in max plus algebra is limited. From (2) we can write a sequence q-step output as follows:

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ \vdots \\ Y(q-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{q-1} \end{bmatrix} X(0) \oplus \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ CB & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ CAB & CB & \varepsilon & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \varepsilon\cdots & \varepsilon \\ CA^{q-2}B & CA^{q-3} & \cdots & CAB & CB \end{bmatrix} \begin{bmatrix} U(0) \\ U(1) \\ U(2) \\ \vdots \\ U(q-1) \end{bmatrix} \quad (7)$$

From (7) we can write the notation of the output sequence simpler, that is $Y_q = \begin{bmatrix} Y(0) & Y(1) & Y(2) & \cdots & Y(q-1) \end{bmatrix}^T$, $U_q = \begin{bmatrix} U(0) & U(1) & U(2) & \cdots & U(q-1) \end{bmatrix}^T$. We can also obtain-step observability matrix, $O_q = \begin{bmatrix} C & CA & CA^2 & \cdots & CA^{q-1} \end{bmatrix}^T$ and matrix

$$H_q = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \\ CB & \varepsilon & \varepsilon & \cdots & \varepsilon \\ CAB & CB & \varepsilon & \cdots & \varepsilon \\ \vdots & \vdots & \cdots & \ddots & \varepsilon \\ CA^{q-2}B & CA^{q-3}B & \cdots & CAB & CB \end{bmatrix}$$

So equation (7) can write in the different way as follows:

$$Y(q) = O_q \otimes X(0) \oplus H_q \otimes U_q \qquad (8)$$

With the same recursively way, from (1) and (2) we obtain:

$$\begin{bmatrix} Y(k) \\ Y(k+1) \\ Y(k+2) \\ \vdots \\ Y(k+q-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{q-1} \end{bmatrix} X(k) \oplus \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ CB & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ CAB & CB & \varepsilon & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \varepsilon \cdots & \varepsilon \\ CA^{q-2}B & CA^{q-3} & \cdots & CAB & CB \end{bmatrix} \begin{bmatrix} U(k) \\ U(k+1) \\ U(k+2) \\ \vdots \\ U(k+q-1) \end{bmatrix} \qquad (9)$$

From (9) we can write the notation of the output sequence simpler, that is $Y_q = \begin{bmatrix} Y(k) & Y(k+1) & \cdots & Y(k+q-1) \end{bmatrix}^T$, $U_q = \begin{bmatrix} U(k) & U(k+1) & \cdots & U(k+q-1) \end{bmatrix}^T$, we can also obtain-step observability matrix, $O_q = \begin{bmatrix} C & CA & CA^2 & \cdots & CA^{q-1} \end{bmatrix}^T$ and matrix

$$H_q = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon \\ CB & \varepsilon & \varepsilon & \cdots & \varepsilon \\ CAB & CB & \varepsilon & \cdots & \varepsilon \\ \vdots & \vdots & \cdots & \ddots & \varepsilon \\ CA^{q-2}B & CA^{q-3}B & \cdots & CAB & CB \end{bmatrix}$$

So equation (9) can write in the different way as follows:

$$Y(q) = O_q \otimes X(k) \oplus H_q \otimes U_q \qquad (9)$$

To start the discussion, we define the output of the system as the observation output that can be explain next.

Definition 8. [3] The observation output $Y(q) \in \mathbb{R}^{m \times q}$ is the output which is given by $Y(q) = O_q \otimes X(k) \oplus H_q \otimes U_q$ with $U_q \in \mathbb{R}_{\max}^{p \times (q-1)}$ and $X(0) \in \overline{\mathbb{R}}_{\max}^{n}$.

Gathering all of the output sequence, we will be directed to the next definition.

Definition 9. The set of Observable output sequence [3].

Let be given a positive integer p and $U_q \in \mathbb{R}_{\max}^{p \times (q-1)}$ is an input sequence, then $\sum_{q.U_q} = \{Y_q \in \mathbb{R}^{m \times q} : Y(q) = O_q \otimes X(k) \oplus H_q \otimes U_q$ with $X(0) \in \overline{\mathbb{R}}_{\max}^{n}$ is the set of observable output sequence.

Considering the necessary and sufficient condition, we can find whether an output sequence is an observable output.

Theorem 3. Given a sequence $Y(q) \in \mathbb{R}^{m \times q}$ and an input sequence $U_q \in \mathbb{R}_{\max}^{p \times (q-1)}$ then $Y(q) \in \sum_{q.Uq}$ if and only if

$$O_q \otimes (-O_q^T \otimes' Y_q) \oplus H_q \otimes U_q = Y_q \qquad (10)$$

Proof. The proof is similar in nature and with the proof of Theorem 2.1.

Definition 10. Latest Event-Time State [3]. Given a q-length sequence of observed outputs Y_q, with a sequence of inputs U_q**,** the latest event-time state $\gamma(k)$ which results in Y_q is

$$\gamma(k) = \max_{X(0)}\{X(k) \in \overline{\mathbb{R}}^n_{\max} : Y_q = O_q \otimes X(k) \oplus H_q \otimes U_q\} \tag{11}$$

where the max is over each component.

Because the latest event-time state should be infinite, then $\gamma(k)$ define to be in $\overline{\mathbb{R}}^n_{\max}$. This infinite output sequence state does not give any information about the systems state. So, we define e finite latest event-time state of the systems, which direct to the definition of weakly observability.

Definition 11. Q-step Weakly Observable [3]. A system is q-step weakly observable if for any q-length sequence of observed outputs $Y_q \in \sum_{q.Uq}$, the latest event-time state $\gamma(k)$ is finite and can be computed from Y_q.

A necessary and sufficient condition for a system to q-step weakly observable is given next.

Theorem 4. [3] A system is q-step weakly observable if only if O_q column astic.

Proof. If O_q is column astic, then Y_q is finite $\gamma(k) = -O_q^T \otimes' Y_q$ is finite too. For every $Y_q \in \sum_{q.Uq}$, with Theorem 3.1 we get that $O_q \otimes (-O_q^T \otimes' Y_q) \oplus H_q \otimes U_q = Y_q$. Furthermore $\gamma(k)$ is an observable output sequence in the other side, if system is q-step weakly observable. The final state should be finite and can computed from the observable output sequence Y_q. By Theorem 3.1 we obtain $O_q \otimes (-O_q^T \otimes' Y_q) \oplus H_q \otimes U_q = Y_q$. So in order $\gamma(k) = -O_q^T \otimes' Y_q$ to be finite, matrix O_q should be column astic.

As continues variable, in max plus algebra we also have the duality of weakly reachable and weakly observable. Duality means that the property of weakly reachable can found from the property of weakly observable, vice versa.

Example: Given a system with matrix:

$$A = \begin{pmatrix} \varepsilon & \varepsilon & 0 \\ 3 & \varepsilon & 2 \\ \varepsilon & 0 & \varepsilon \end{pmatrix}, B = \begin{pmatrix} 0 \\ 2 \\ \varepsilon \end{pmatrix} \text{ dan } C = \begin{pmatrix} \varepsilon & 0 & \varepsilon \end{pmatrix}$$

We will investigate the property of system, whether its weakly reachable or weakly observable ?

1) Reachability matrix at 2-step, we obtain:

$$\Gamma_2 = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} B & A \otimes B \end{bmatrix} = \begin{pmatrix} 0 & \max(\varepsilon+0, \varepsilon+2, 0+\varepsilon) \\ 2 & \max(3+0, \varepsilon+2, 2+\varepsilon \\ \varepsilon & \max(\varepsilon+0, 0+2, \varepsilon+\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ 2 & 3 \\ \varepsilon & 2 \end{pmatrix}$$

Shown that at 2-step, the system is weakly reachable. For 3-step, we obtain the reachability matrix as:

$$\Gamma_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} B & A \otimes B & A^{\otimes 2} \end{bmatrix}$$

$$= \begin{pmatrix} 0 & \max(\varepsilon+0, \varepsilon+3, \varepsilon+2) & \max(\varepsilon+\varepsilon, \varepsilon+3, 0+2) \\ 2 & \max(3+0, \varepsilon+2, 2+\varepsilon) & \max(\varepsilon+\varepsilon, \varepsilon+3, 0+2) \\ \varepsilon & \max(3+0, \varepsilon+2, 2+\varepsilon) & \max(\varepsilon+3, 3+\varepsilon, 2+2) \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon & 2 \\ 2 & 3 & 4 \\ \varepsilon & 2 & 3 \end{pmatrix}$$

For 4-step, we obtain the reachability matrix as:

$$\Gamma_4 = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} B & A\otimes B & A^{\otimes 2}\otimes B & A^{\otimes 3B} \end{bmatrix}$$
$$= \begin{pmatrix} 0 & \max(\varepsilon+0,\varepsilon+2,0+\varepsilon) & \max(\varepsilon+\varepsilon,\varepsilon+3,0+2) & \cdots \\ 2 & \max(3+0,\varepsilon+2,2+\varepsilon) & \max(\varepsilon+3,3+\varepsilon,2+2) & \cdots \\ \varepsilon & \max(\varepsilon+0,0+2,\varepsilon+\varepsilon) & \max(\varepsilon+\varepsilon,0+3,\varepsilon+2) & \cdots \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon & 2 & 3 \\ 2 & 3 & 4 & 5 \\ \varepsilon & 2 & 3 & 4 \end{pmatrix}$$

Shown that for 3-step, 4-step and so on the system is always weakly reachable.

2) Observability matrix at 2-step, we obtain:

$$O_2 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \max(\varepsilon+\varepsilon,0+3,\varepsilon+\varepsilon) & \max(\varepsilon+\varepsilon,0+\varepsilon,\varepsilon+\varepsilon) & \max(\varepsilon+0,0+2,\varepsilon+\varepsilon) \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ 3 & \varepsilon & 2 \end{pmatrix}$$

Shown that at 2-step, the system is weakly observable. For 3-step, we obtain the robservability matrix as:

$$O_3 = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \max(\varepsilon+\varepsilon,0+3,\varepsilon+\varepsilon) & \max(\varepsilon+\varepsilon,0+\varepsilon,\varepsilon+\varepsilon) & \max(\varepsilon+0,0+2,\varepsilon+\varepsilon) \\ \max(3+\varepsilon,\varepsilon+3,2+\varepsilon) & \max(3+\varepsilon,\varepsilon+\varepsilon,2+0) & \max(3+0,\varepsilon+2,2+\varepsilon) \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ 3 & \varepsilon & 2 \\ \varepsilon & 2 & 3 \end{pmatrix}$$

For 4-step, we obtain the observability matrix as:

$$O_4 = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \max(\varepsilon+\varepsilon,0+3,\varepsilon+2) & \max(\varepsilon+0,0+\varepsilon,\varepsilon+\varepsilon) & \max(\varepsilon+0,0+2,\varepsilon+\varepsilon) \\ \max(3+\varepsilon,\varepsilon+\varepsilon,2+\varepsilon) & \max(3+\varepsilon,\varepsilon+\varepsilon,2+0) & \max(3+0,\varepsilon+2,2+\varepsilon) \\ \max(\varepsilon+\varepsilon,2+3,3+\varepsilon) & \max(\varepsilon+\varepsilon,2+\varepsilon,3+0) & \max(\varepsilon+0,2+2,2+\varepsilon) \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ 3 & \varepsilon & 2 \\ \varepsilon & 2 & 3 \\ 5 & 3 & 4 \end{pmatrix}$$

Shown that for 3-step, 4-step and so on the system is always weakly observable.

From 1) and 2) we get that the system had weakly reachable and weakly observable since 2-step. Because Γ_2 is row astic and O_2 is column astic. Matrix Γ_3 and Γ_4 also row astic, matrix O_3 and O_4 are column astic too. We can conclude, for step-q with $q \geq 2$ the system should be weakly reachable and weakly observable.

4. CONCLUTION AND FUTURE WORK

From the discussion, we obtain that the necessary and sufficient condition of weakly reachable is the row astic of its reachability matrix. Then the necessary and sufficient condition of weakly observable is the column astic of its observability matrix. If at q-step the system is weakly reachable or weakly observable, then for step-(q+1), step-(q+2), and so on the system will should be weakly reachable or weakly observable. For the future work, the discussion could be explored the strongly observable and reachable of the system. And can to determine for finite step for system is weakly reachable or weakly observable.

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