On The Structural Properties of Latin Square in Max-Plus Algebra

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***Abstract*— Matrices *L* of size *n* × *n* are called Latin square if every column and every row of *L* contain *n* different numbers. And, Max-Plus Algebra is algebraic system using two operations, *max* and *plus*. In this paper, we derive some properties of a Latin square in Max-Plus Algebra and their eigenvalues and corresponding eigenvector.**

***Index Terms*— Latin square, Max-Plus Algebra, Eigenvalue, Eigenvector.**

I. Introduction

A Latin square of order n is square matrix of size $n \times n$ such that every row and every column has n distinct numbers. For convenience, we use $\underline{n} = \{1,2,\ldots,n\}$. The notion of Latin square is firstly introduced by Leonhard Euler. A Latin square is in *reduced form* if first row is $[1, 2, 3, \ldots, n]$ and first column is $[1, 2, 3, \ldots, n]^T$. If numbers in both diagonals also distinct then we called it by Latin square-X. An example of Latin square and reduced Latin square is given below

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

The study of Latin square is mainly about discrete mathematics aspect especially enumeration of Latin square. Until now, the exact number of Latin square is known only for $1 \le n \le 11$. The result of enumeration Latin square-X is can be found in [1]. The number of Latin square of order 5 and 6 is 960 and 92160 respectively, and for order 7 the number of Latin square is increasing sharply, that is 862848000.

Accordance with its name, Max-Plus Algebra is algebra that using two operations, *max* and *plus*. In Max-Plus algebra defined algebraic structure $(R_\varepsilon, \oplus, \otimes)$ where R_ε is set of extended real numbers, i.e. $R_\varepsilon = R \cup \{-\infty\}$. In this paper, we denoted infinite element, i.e. $\varepsilon = -\infty$ Operation *max* denoted by $\oplus$ and defined by $a \oplus b = \max\{a, b\}$, and operation *plus* denoted by $\otimes$ and denoted by $a \otimes b = a + b$ for every a, b in R_ε. For example, $3 \oplus 2 = \max\{3,2\} = 3$ and $-2 \otimes 6 = -2 + 6 = 4$.

It is easy to show that both operations $\oplus, \otimes$ are commutative in max-plus algebra. Because all $x \in R_\varepsilon$ satisfy $x \oplus \varepsilon = \varepsilon \oplus x = x$ and $x \otimes 0 = 0 \otimes x = x$, then the zero and unit element in max-plus algebra is ε and 0, respectively.

The set of all $n \times m$ matrices in max-plus algebra is denoted by $R_\varepsilon^{n \times m}$, and for $m = 1$ we denoted the set of all $n \times 1$ vectors by R_ε^n. Let $A \in R_\varepsilon^{n \times m}$, the entry of A in i^{th} row and j^{th} column is denoted by $a_{i,j}$ and sometimes we write $[A]_{i,j}$. The i^{th} row and j^{th} column of A is denoted by $[A]_{i,-}$ and $[A]_{-,j}$ respectively. For $A, B \in R_\varepsilon^{n \times m}$ $A \oplus B$ is defined by

$$[A \oplus B]_{i,j} = a_{i,j} \oplus b_{i,j} = \max\{a_{i,j}, b_{i,j}\}$$

and for $A \in R_\varepsilon^{n \times p}$, $B \in R_\varepsilon^{m \times p}$, $A \otimes B$ is defined by

$$[A \otimes B]_{i,j} = \max\{(a_{i,1} + b_{1,j}), (a_{i,2} + b_{2,j}), \ldots, (a_{1,p} + b_{p,j})\}$$

For example,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ \varepsilon & -1 & 4 \\ 0 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & \varepsilon \\ 0 & -5 & 1 \\ -1 & \varepsilon & 2 \end{bmatrix}$$

We get

$$A \oplus B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 2 & 3 \end{bmatrix}, A \otimes B = \begin{bmatrix} 4 & 0 & 4 \\ 3 & -6 & 6 \\ 3 & -1 & 5 \end{bmatrix}$$

In max-plus algebra, we defined $A^{\otimes 2} = A \otimes A$ or generally $A^{\otimes k+1} = A \otimes A^{\otimes k}$ for $k = 1,2,\ldots$

Let $A \in R_\varepsilon^{n \times n}$, a digraph (directed graph) of A is denoted as $G(A)$. Graph $G(A)$ has n vertices and there is an edge from vertex i to vertex j if $a_{j,i} \ne \varepsilon$ and this edge is denoted by (i, j). Weight of edge (i, j) is denoted by $w(i, j)$ and equal to $a_{j,i}$. Sequence of edges $(j_1, j_2), (j_2, j_3), \ldots, (j_{k-1}, j_k)$ is called by a *path* and if all vertices $j_1, j_2, j_3, \ldots, j_{k-1}$ are different then called by *elementary path*. Circuit is an *elementary close path*, i.e. $(j_1, j_2), (j_2, j_3), \ldots, (j_{k-1}, j_1)$. Circuit that consisting of single edge, from a vertex to itself, is called by *looping*.

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Weight of a path $p = (j_1, j_2), (j_2, j_3), \ldots, (j_{k-1}, j_k)$ is denoted by $|p|_w$ and equal to the sum of weight each edge. Length of path p is denoted by $|p|_l$ and equal to the number of edges in p. Average weight of p is defined by $\frac{|p|_w}{|p|_l}$.

Any circuit with maximum average weight is called by *critical circuit.* Graph *G(A)* is called strongly connected if there is path for any vertex *i* to any vertex *j* in *G(A)*. If graph *G(A)* is strongly connected, then matrix *A* is *irreducible.* From matrix *A*, $[A^{\otimes k}]_{i,j}$ is equal to the maximal weight of a path with length *k* from vertex *i* to vertex *j*.

II. LATIN SQUARE IN MAX-PLUS ALGEBRA

Because the discussion is in max-plus algebra, it is allowed to use infinite element $\varepsilon = -\infty$ as number/element of Latin square. In this paper we define two types of Latin square:

a. Latin square without infinite element, the numbers that used are in $\underline{n} = \{1, 2 \ldots, n\}$
b. Latin square with infinite element, the numbers that used are in $\underline{n}_\varepsilon = \{-\infty, 1, 2 \ldots, n-1\}$

The set of all Latin squares of order *n* without infinite element is denoted by LS^n and the set of all Latin squares of order *n* with infinite element is denoted by LS^n_ε. Example of two types of Latin square is given below.
We can infer that $L_1 \in LS^4$ and $L_2 \in LS^4_\varepsilon$.

$$L_1 = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 1 & 4 & 2 \\ 4 & 2 & 3 & 1 \\ 1 & 4 & 2 & 3 \end{bmatrix}, L_2 = \begin{bmatrix} \varepsilon & 1 & 3 & 2 \\ 3 & 2 & \varepsilon & 1 \\ 2 & \varepsilon & 1 & 3 \\ 1 & 3 & 2 & \varepsilon \end{bmatrix}$$

III. PROPERTIES OF LATIN SQUARE IN MAX-PLUS ALGEBRA

Properties of Latin square in max-plus algebra that will be derived are:

a. Irreducible. Are all Latin squares in max-plus algebra irreducible?
b. Close under operation $\oplus$. Are all Latin squares closed under operation $\oplus$?
c. Close under operation $\otimes$. Are all Latin squares closed under operation $\otimes$?

A. Property of Irreducibility

Lemma 1. *All Latin squares are irreducible matrix.*

Proof.
Let L be Latin square. If $L \in LS^n$ then all numbers of L are finite. Therefore, in graph $G(L)$ there is a path with length 1 from vertex *i* to vertex *j* for all $i, j \in \underline{n}$. Then we can conclude that $G(L)$ is strongly connected and consequently L is irreducible.

If $L \in LS^n_\varepsilon$ we consider matrix $L^{\otimes 2} = L \otimes L$. Because there is only one ε in every row and every column of L then $[L^{\otimes 2}]_{i,j}$ is finite for all $i, j \in \underline{n}$. Therefore in graph $G(L)$ there are some paths with length at least 2 from vertex *i* to vertex *j* for all $i, j \in \underline{n}$. Then we can conclude that $G(L)$ is strongly connected and consequently L is irreducible.

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B. *Property of closed under operation* $\oplus$

We say that Latin squares are closed under operation $\oplus$ if for all Latin squares *A* and *B*, $A \oplus B$ is Latin square.

Lemma 2. Let both *A* and *B* are in LS^n or in LS^n_ε. $A \oplus B$ is Latin square if and only if $A = B$.

Proof.
Let $A, B \in LS^n$ and $C = A \oplus B$. Because $[A]_{i,j}, [B]_{i,j} \in \underline{n}$ then $[C]_{i,j} = [A]_{i,j} \oplus [B]_{i,j} \in \underline{n}$. If C is Latin square then $C \in LS^n$. To prove $A = B$ we only need considering first column. See the illustration below

$$\begin{bmatrix} c_{1,1} \\ c_{2,1} \\ . \\ . \\ c_{n,1} \end{bmatrix} = \begin{bmatrix} a_{1,1} \oplus b_{1,1} \\ a_{2,1} \oplus b_{2,1} \\ . \\ . \\ a_{n,1} \oplus b_{n,1} \end{bmatrix}$$

Let $a_{1,1} = x \in \underline{n}$, if $b_{1,1} \neq x$ then x can appear more than one or not appear in left side matrix. Therefore we get $a_{1,1} = b_{1,1}$ and by same way we get $a_{2,1} = b_{2,1}, \ldots, a_{n1} = b_{n,1}$ or generally $a_{i,1} = b_{i,1}$ for all $i \in \underline{n}$. Consequently, the first column of *A* and *B* is equal or generally $[A]_{-,i} = [B]_{-,i}$ for all $i \in \underline{n}$, in other word $A = B$.

Conversely, if $A = B$ then

$$c_{i,j} = a_{i,j} \oplus b_{i,j} = a_{i,j} \oplus a_{i,j} = \max\{a_{i,j}, a_{i,j}\} = a_{i,j}$$

Consequently, $C = A \oplus B = A \oplus A = A$ and C is Latin square.
For $A, B \in LS^n_\varepsilon$ it can be proved by similar way.

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By Lemma 2 we can conclude that Latin square is not closed under operation $\otimes$

Example:

$$A=\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}, B=\begin{bmatrix}2&1&3\\1&3&2\\3&2&1\end{bmatrix}$$

We get

$$A\oplus B=\begin{bmatrix}2&2&3\\2&3&2\\3&2&2\end{bmatrix}$$

and $A \oplus B$ is not Latin square.

C. Property of closed under operation $\otimes$

We say that Latin squares are closed under operation $\otimes$ if for all Latin square *A* and *B*, $A \otimes B$ is Latin square. In other word, there is Latin square C and natural number p such that $A \otimes B = p \otimes C$.

Lemma 3. If three Latin squares $A, B, C \in LS^n$ satisfy $A \otimes B = p \otimes C$ then $p = n$.

Proof. Let $A \otimes B = D$, and D is Latin square, then

$$[D]_{i,j} = \max\{(a_{i,1} + b_{1,j}), (a_{i,2} + b_{2,j}),..., (a_{1,n} + b_{n,j})\}$$

Because maximum value both $a_{i,k}$ and $a_{k,j}$ for all $k \in \underline{n}$ are n , then maximum value of $[D]_{i,j}$ is $2n$. Next, we determine the minimum value of $[D]_{i,j}$.

Let $a_{i,k} + b_{k,j} = d_k$ then $[D]_{i,j} = \max\{d_1, d_2,..., d_n\}$ and we know that $\sum_{k=1}^{n} d_k = n(n+1)$. It is easy to find that the minimum value of $[D]_{i,j}$ occur when

$$d_1 = d_2 = ... = d_n = n + 1,$$

then $[D]_{i,j} = \max\{n+1, n+1,..., n+1\} = n + 1$. If there are some k such that $d_k < n+1$ then there are some l such that $d_l > n+1$ and consequently $[D]_{i,j} > n + 1$. So, it is clear that minimum value of $[D]_{i,j}$ is $n + 1$.

Because D is Latin square of order n and $n + 1 \le [D]_{i,j} \le 2n$, then we can conclude that

$$[D]_{i,j} \in \{n+1, n+2,..., 2n-1, 2n\} = \{n + k \mid \forall k \in \underline{n}\}$$

So, if $D = p \otimes C$, we get $p = n$.

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From Lemma 3, one of requirement for $A \otimes B$ producing Latin square is for all $i \in \underline{n}$ there is $j \in \underline{n}$ such that

$$[A]_{i,-} + [B]_{-,j} = [n+1 \quad n+1 \quad . \quad . \quad n+1 \quad n+1]$$

So we can conclude that Latin square is not closed under operation $\otimes$.

Example:

$$A=\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}, B=\begin{bmatrix}2&3&1\\3&1&2\\1&2&3\end{bmatrix}$$

We get

$$A\otimes B=\begin{bmatrix}5&5&6\\5&6&5\\6&5&5\end{bmatrix}$$

and $A \otimes B$ is not Latin square.

IV. EIGENPROBLEM OF LATIN SQUARE IN MAX-PLUS ALGEBRA

Matrix L of order n has eigenvalue $\lambda \in R$ and corresponding eigenvector $v \in R_{\varepsilon}^{n\times n}$ if both of them satisfy

$$L \otimes v = \lambda \otimes v$$

In this paper we denoted $\lambda(A)$ be eigenvalue of matrix *A*. From [2,3], there is algorithm to find eigenvalue corresponding eigenvector that called by *Power Algorithm*. If L is irreducible matrix, then eigenproblem is equivalent to problem to find critical circuit in *G*(*L*), where eigenvalue is equal to weight of that critical circuit.

We define $L_{\lambda} = (-\lambda) \otimes L$ and

$$L_{\lambda}^{+} = L \oplus L^{\otimes 2} \oplus L^{\otimes 3} \oplus ... \oplus L^{\otimes n}$$

It can be proved that $[L_{\lambda}^{+}]_{-,j}$ is eigenvector of L if $[L_{\lambda}^{+}]_{j,j} = 0$ [3].

A. Eigenvalue of Latin square in Max-Plus Algebra

From Lemma 1, all Latin squares are irreducible matrix. Therefore, to find eigenvalue of *L* we need to find the weight of critical circuit in *G*(*L*).

If $L \in LS^n$ then $[L]_{j,j} \in \{1,2,..., n\}$ and it is clear that $\max\{[L]_{i,j}\} = n$. Let

$$p = (j_1, j_2), (j_2, j_3),..., (j_{k-1}, j_k), (j_k, j_1)$$

be critical circuit with length k with $k \le n$ in *G*(*L*), then

$$| p |_w = w(j_1, j_2) + w(j_2, j_3) + ... + w(j_{k-1}, j_k) \le kn$$

and average weight of p is equal to

$$\frac{| p |_w}{| p |_l} \le \frac{kn}{k} = n$$

Because n occurs exactly one in every row and column of *L*, we can ensure that average weight of critical circuit p is equal to *n*. Therefore, eigenvalue of *L* is equal to *n*, in other word $\lambda = n$.

By the same method, we get eigenvalue of $L \in LS_{\varepsilon}^{n}$, that is $\lambda = n - 1$.

B. Eigenvector of Latin square in Max-Plus Algebra

Let $A \in LS^n$ and $B \in LS_{\varepsilon}^{n}$. From the definition we get $A_{\lambda} = (-n) \otimes A$ and $B_{\lambda} = (-n+1) \otimes A$. It is clear that average weight of critical circuit both in $G(A)$ and and $G(B)$ is 0.

From [3], if p is critical circuit of $G(L)$ then for all vertices η in p satisfy $[L_\lambda^+]_{\eta,\eta} = 0$. But in this case, for $A \in LS^n$ and $B \in LS_\varepsilon^n$, the average weight of A and B is equal to maximum value of matrix A and B, i.e

$$\lambda(A) = \max\{[A]_{i,j}\} = n$$

and

$$\lambda(B) = \max\{[B]_{i,j}\} = n-1.$$

Moreover, n occurs exactly one in every row and column of A and $n-1$ occurs exactly one in every row and column of B. Consequently, there is always critical circuit in A such that the initial vertex is *i*. The same condition is for matrix B. Therefore, for all $\eta \in \underline{n}$ in satisfy

$$[A]_{\eta,\eta} = 0 \text{ and } [B]_{\eta,\eta} = 0$$

or in other word, all diagonal entries of both A and B are equal to 0. So we can conclude that all columns of A_λ^+ is eigenvector of A and all columns of B_λ^+ is eigenvector of B

Example.

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

We get $\lambda(A) = 4$ and

$$A_\lambda = \begin{bmatrix} -3 & -1 & 0 & -2 \\ -2 & 0 & -1 & -3 \\ 0 & -2 & -3 & -1 \\ -1 & 3 & -2 & 0 \end{bmatrix}$$

$$A_\lambda^+ = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & -2 \\ 0 & -1 & 0 & -1 \\ -1 & -2 & -1 & 0 \end{bmatrix}$$

from A_λ^+ we get three different columns and we can check that all of them are eigenvector of A

$$A \otimes \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \end{bmatrix} = 4 \otimes \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix},$$

$$A \otimes \begin{bmatrix} -1 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 2 \end{bmatrix} = 4 \otimes \begin{bmatrix} -1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

$$A \otimes \begin{bmatrix} -1 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 4 \otimes \begin{bmatrix} -1 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

V. CONCLUSION

In this paper we can conclude that Latin squares have properties in Max-Plus Algebra, that is

1. All Latin squares are irreducible matrix
2. Latin squares are not closed under operation $\oplus$
3. Latin squares are not closed under operation $\otimes$

Moreover, eigenvalue of Latin squares L are equal to maximum value of L and all columns of L_λ^+ are eigenvector of L.

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