A Study Permutation Theory and Its Application to Enumeration of Latin Square-X

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Abstract. A Latin square of order n is an array or matrix of size $n \times n$ where in each row and column contains n different numbers or symbols. Enumeration of Latin square is not easy problem even using a computer. Until now, the exact number of Latin square is known only for $1 \le n \le 11$. The basic in Latin square is permuted n numbers. By considering the characteristic of permutation that appears in Latin square-x, in this paper we will discuss about theory of permutation about Latin square and then applying its to enumeration of Latin square-x.

Keyword:Latin square, Enumeration, Latin square-x, Permutation.

1 Introduction

Enumeration is a problem that associated with counting. In combinatorics, enumeration means determining the exact number of elements of finite sets. For example, established basket ball team consist of 5 players from 7 candidate players. The number teams that can be established are 21.

Latin square of order n is an array or matrix size $n \times n$ with n symbols such that in each row and column filled by the permutation of symbols [4], in other word the entries in each row and in each column are distinct [5]. Latin square firstly introduce by Swiss mathematician, Leonhard Euler. The study of Latin square has long tradition in combinatorics [6]. A Latin square of order n can be called by Latin square-x (doubly diagonalized) if both its diagonals consist of n distinct symbols [3]. An example of Latin square-x is shown in Fig. 1.

Enumeration of Latin square is not easy problem even using computer. Until now, the exact number of Latin square is known only for order $1 \le n \le 11$[1, 2, 4].

2	3	1	4
1	4	2	3
4	1	3	2
3	2	4	1

Fig. 1. Latin Square-X of Order 4

The notion of permutation is related to the act of rearranging objects or values. A permutation of a set of objects is an arrangement of those objects into a particular order. For example there are six permutations from element of the set $\{1, 2, 3\}$, that is (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2) and (3,2,1). For simply, we write a permutation without parentheses and commas. So we will write 123 rather than (1, 2, 3).

In algebra, especially group theory, permutation is a bijective mapping on set X. A family of all permutations from X is called by symmetry group S_X [8], if $X = \{1, 2, 3 \ldots, n\}$ we write S_n rather than S_X. Let $i_1 i_2 \ldots i_n$ be a permutation from $X = \{1, 2, \ldots, n\}$ and defines a function $\alpha : X \to X$ as $\alpha(1) = i_1, \alpha(2) = i_2, \ldots, \alpha(n) = i_n$. For $i \in \{1, 2, \ldots, n\}$ such that $\alpha(i) = i$, then i fixed by α.

Permutation matrix is a square matrix such that in each column and row there is exact one entry 1 and the others is 0. The example of permutation matrix is shown in Fig. 2.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 2. Permutation Matrix of Order 4

Let a_{ij} be entry of permutation matrix at i^{th} row and j^{th} column, from matrix A we can get a permutation function α. If $a_{ij} = 1$, then $\alpha(i) = j$. From A we get $a_{12} = a_{24} = a_{31} = a_{43} = 1$, then we have $\alpha(1) = 2, \alpha(2) = 4, \alpha(3) = 1$, and $\alpha(4) = 3$ and permutation representation of matrix A is 2413 (see Fig. 2).

From a Latin square order n, we can get n permutation matrix that represent fixed number $i \in \{1, 2, \ldots, n\}$. Let L be matrix from Latin square in Fig. 1 (see Fig. 3), then from L we get four permutation matrices, that is L_1, L_2, L_3 and L_4 (see Fig. 4).

$$L = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 4 & 2 & 3 \\ 4 & 1 & 3 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

Fig. 3. Matrix Representation from Latin Square

We can easily check that $L = L_1 + 2L_2 + 3L_3 + 4L_4$, or generally for order n, $L = L_1 + 2L_2 + \ + nL_n$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be permutation of $L_1, L_2, \ldots, L_n$ respectively, again we can check that

$$\alpha_i(k) \neq \alpha_j(k) \text{ for } i, j \in \{1, 2, \ldots, n\} \text{ and } i \neq j \tag{1}$$

$$L_1 = \begin{bmatrix} 0\,0\,1\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 0\,1\,0\,0 \end{bmatrix}$$

3124 1342

$$L_3 = \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \\ 1\,0\,0\,0 \end{bmatrix} \qquad L_4 = \begin{bmatrix} 0\,0\,0\,1 \\ 0\,1\,0\,0 \\ 1\,0\,0\,1 \\ 0\,0\,1\,0 \end{bmatrix}$$

2341 4213

Fig. 4. Permutations matrix from Latin Square and its permutation representation

2 Latin Square Transformation

From a Latin square L, we can get another Latin square L' by the rule below:

- From Latin square L of order n we get n different permutation matrices, that is L_1, L_2,. . ., L_n.
- From n permutation matrices we get n different permutation representation
- The i^{th} row of L' will be filled by permutation representation of matrix L_i for $i = 1, 2, \ldots, n$.

For example, the permutation representation of L_1, L_2, L_3 and L_4 from Fig. 4 is 3124, 1342, 2431 and 4213 respectively, and then these permutations will be filled to 4 x 4 array and get another Latin square, of course uniquely.

$$\begin{array}{|c|c|c|c|} \hline 2&3&1&4 \\ \hline 1&4&2&3 \\ \hline 4&1&3&2 \\ \hline 3&2&4&1 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|c|} \hline 3&1&2&4 \\ \hline 1&3&4&2 \\ \hline 2&4&3&1 \\ \hline 4&2&1&3 \\ \hline \end{array}$$

Fig. 5. Latin Square Transformation

It can be checked that after Latin square transformation thrice, we get initial Latin square.

3 Method

3.1 Permutations from Latin Square-X

From number $1, 2, \ldots, n$ in a Latin square we can get n different permutation matrices of order n. Let a_{ij} be entry of permutation matrix at i^{th} row and j^{th} column, then main diagonal entry is a_{ii} and secondary diagonal entry is a_{ij} with $i + j = n + 1$.

We know that in permutation matrix, if $a_{ij} = 1$, then a permutation function α satisfy $\alpha(i) = j$. Because the entries at main diagonal and secondary diagonal are distinct, we can conclude that permutation from Latin square-x i.e α, satisfy:

i. There is exactly one i such that $\alpha(i) = i$.
ii. There is exactly one j such that $\alpha(j) + j = n + 1$. (2)

From (2) i we know that there is exactly one $i \in \{1, 2, \ldots, n\}$ that fixed by α.

3.2 Partition of Permutation

Let P_n be a set all permutation from Latin square-x of order n. P_n will be partitioned into subsets namely $P_{n,i:j}$ based on i, j that satisfy (2). For example $1423 \in P_{4,1:3}$ because $\alpha(1) = 1, \alpha(3) + 3 = 5$ and there is no other $i, j \in \{1, 2, 3, 4\}$ satisfy (2).

For n even, P_n can be partitioned into $n(n-2)$ subsets and for n odd, P_n can be partitioned into $(n-1)^2$ subsets. For example

- $P_4 = \{P_{4,1:2}, P_{4,1:3}, P_{4,2:1}, P_{4,2:4}, P_{4,3:1}, P_{4,3:4}, P_{4,4:2}, P_{4,4:3}\}$.
- $P5 = \{P_{5,1:2}, P_{5,1:4}, P_{5,2:1}, P_{5,2:5}, P_{5,3:3}, P_{4,4:1}, P_{5,4:5}, P_{5,5:2}, P_{5,5:4}\}$.

3.3 Enumeration of Permutation

We know that all permutation in P_n satisfy (2). Because of partitioning of P_n into $P_{n,i:j}$, to enumerate all permutations from Latin square-X is equal to enumerate the cardinality of each $P_{n,i:j}$ or $|P_{n,i:j}|$. For n even or n odd with $i \neq j$, it is easy to check that the value of $|P_{n,i:j}|$ is not depend i and j, so $|P_{n,i:j}|$ is the same for all i, j. For n odd with $i = j$, the value of $|P_{n,i:i}|$ is higher than others. The value of $|P_n|$ is equal to the sum all $|P_{n,i:j}|$. To enumerate all permutation in P_n is used algebra software GAP (*Group*, *Algorithm*, *Programming*) [9].

For example,

- $P_4 = \{1342, 1423, 4213, 3241, 4132, 2431, 2314, 3124\}$
- $P_5 = \{14253, 14532, 13524, 15423, 52134, 52413, 32451, 42531, 21354, 25314, 41352, 45312, 51243, 53142, 23541, 35241, 24135, 34215, 31425, 43125\}$

3.4 Enumeration of Latin Square-X

It has been explained that by Latin square transformation, we get other Latin square uniquely and all permutations from Latin square-x satisfy (1). Because the uniquely of Latin square from Latin square transformation, the enumeration problem of all possible Latin square-x is equal to problem of counting "how to select and arrange" n permutations from P_n to $n \times n$ array that produce Latin square. This problem transformation made difficulty level of enumeration of Latin square-x decrease.

Because P_n has been partitioned into $P_{n,i:j}$, we need to consider value of i, j such that the n permutations that we selected can produce Latin square.

Of course we cant select two permutations from the same $P_{n,i:j}$, because the number at position i, j is the same. For example, both 14253 and 14532 are permutation from $P_{5,1:2}$, the number at 1st and 2nd position are the same, that is 1 and 4 respectively. Then from n permutations, the index i, j each $P_{n,i:j}$ are all different.

We defined *collection of permutation* Kn that can produce Latin square. For example,

$$K4 = \{\{P_{4,1:2}, P_{4,2:1}, P_{4,3:4}, P_{4,4:3}\}, \{P_{4,1:2}, P_{4,2:4}, P_{4,3:1}, P_{4,4:3}\}, \{P_{4,1:3}, P_{4,2:1}, P_{4,3:4}, P_{4,4:2}\}, \{P_{4,1:3}, P_{4,2:4}, P_{4,3:1}, P_{4,4:2}\}\}.$$

Then we have four possible collection of permutation for order 4, or we can write $|K4| = 4$. Using GAP, we get $|K5| = 4, |K6| = |K7| = 8, |K8| = |K9| = 4752$. We will enumerate the number of Latin square-x manually for order 4, for order 5, 6 and 7 we enumerate using algebra software GAP.

For order 4 we have $P_4 = \{1342, 1423, 4213, 3241, 4132, 2431, 2314, 3124\}$ and $K4 = \{\{P_{4,1:2}, P_{4,2:1}, P_{4,3:4}, P_{4,4:3}\}, \{P_{4,1:2}, P_{4,2:4}, P_{4,3:1}, P_{4,4:3}\}, \{P_{4,1:3}, P_{4,2:1}, P_{4,3:4}, P_{4,4:2}\}, \{P_{4,1:3}, P_{4,2:4}, P_{4,3:1}, P_{4,4:2}\}\}$. Hence, we have four possible.

i. First collection
First collection is $\{P_{4,1:2}, P_{4,2:1}, P_{4,3:4}, P_{4,4:3}\} = \{1342, 4213, 2431, 3124\}$. By this collection we can produce a Latin square.

1342
4213
2431
3124
→

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

→ Latin Square

ii. Second collection
Second collection is $\{P_{4,1:2}, P_{4,2:4}, P_{4,3:1}, P_{4,4:3}\} = \{1342, 3241, 4132, 3124\}$. By this collection we cannot produce a Latin square.

1342
3241
4132
3124
→

1	3	4	2
3	2	4	1
4	1	3	2
3	1	2	4

→ not Latin Square

iii. Third collection
Third collection is $\{P_{4,1:3}, P_{4,2:1}, P_{4,3:4}, P_{4,4:2}\} = \{1423, 4213, 2431, 2314\}$. By this collection we cannot produce a Latin square.

1423
4213
2431
2314
→

1	4	2	3
4	2	1	3
2	4	3	1
2	3	1	4

→ not Latin Square

iv. Fourth collection
Fourth collection is $\{P_{4,1:3}, P_{4,2:4}, P_{4,3:1}, P_{4,4:2}\} = \{1423, 3241, 4132, 2314\}$. By this collection we can produce a Latin square.

$$\begin{array}{c} 1423 \\ 3241 \\ 4132 \\ 2314 \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1&4&2&3 \\ \hline 3&2&4&1 \\ \hline 4&1&3&2 \\ \hline 2&3&1&4 \\ \hline \end{array} \rightarrow \text{ Latin Square}$$

From four possible collections, we get a Latin square only from the first and the fourth. Then for $n = 4$, the total number of Latin square-x is $2 \times 4! = 48$, because we can arrange rows in 4! ways.

4 Result and Discussion

We know that all permutations from Latin square satisfy (2). The complete result of enumeration of P_n is shown in Table 1. After enumeration of P_n, we can enumerate the number of Latin square-x by chose and arrange n permutations to $n \times n$ array that produce a Latin square, the complete result of enumeration of Latin square-x is shown in Table 2.

Table 1. The number of permutation from Latin square-x [7]

P_n	$\lvert P_{n,i:j}\rvert^*$	$\lvert P_{n,i:j}\rvert^{**}$	$\lvert P_n\rvert$
P_4	1	0	$1 \times 8 = 8$
P_5	2	2	$2 \times 8 + 4 = 20$
P_6	4	0	$4 \times 24 = 96$
P_7	24	80	$24 \times 24 + 80 = 656$
P_8	116	0	$116 \times 48 = 5568$
P_9	920	4752	$920 \times 48 + 4752 = 48912$

$*$: for $i \neq j$,$**$: for $i = j$

Table 2. The number of Latin square-x [7]

n	$L(n)$	Total
4	2	$2 \times 4! = 48$
5	8	$8 \times 5! = 960$
6	128	$128 \times 6! = 92160$
7	171200	$171200 \times 7! = 862848000$

5 Conclusion

After analyzed the permutation from Latin square-x, it can be concluded that all permutation satisfy (2), then using those permutations we can enumerate the number of Latin square-x by chose and arrange n different permutations to $n \times n$ array. If from that we get a Latin square then we have Latin square-x.

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